Chapter 10

Binomial Coefficients

10.1 Basic properties

Recall that $\binom{n}{k}$ is the number of k-element subsets of an n-element set, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}.$$

The quantities $\binom{n}{k}$ are called *binomial coefficients* because of their role in the *Binomial Theorem*, apparently known to the 11th century Persian scholar Omar Khayyam. Before we state and prove the theorem let us consider some important identities that involve binomial coefficients. One that follows immediately from the algebraic definition is

$$\binom{n}{k} = \binom{n}{n-k}.$$

This also has a nice combinatorial interpretation: Choosing a k-element subset B from an n-element set uniquely identifies the complement $A \setminus B$ of B in A, which is an $(n-k)$-subset of A. This defines a bijection between k-element and $(n-k)$-element subsets of A, which implies the identity.

Another relation between binomial coefficients is called *Pascal's rule*, although it was known centuries before Pascal's time in the Middle East and India:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

This can be easily proved algebraically:

$$\begin{aligned}
\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(k-1)!(n+1-k)!}+\frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k!(n+1-k)!}+\frac{n!(n+1-k)}{k!(n+1-k)!} \\
&= \frac{n!k+n!(n+1-k)}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

Pascal's rule also has a combinatorial interpretation: $\binom{n+1}{k}$ is the number of k-element subsets of an n-element set A. Fix an element $a \in A$. A subset of A either contains a or it doesn't. k-element subsets of A that do not contain a are in fact k-element subsets of $A \setminus \{a\}$ and their number is $\binom{n}{k}$. k-element subsets of A that do contain a bijectively correspond to $(k-1)$-element subsets of $A \setminus \{a\}$, the number of which is $\binom{n}{k-1}$. The identity follows.

Another illuminating identity is the *Vandermonde convolution*:

$$\binom{m+n}{l} = \sum_{k=0}^{l}\binom{m}{k}\binom{n}{l-k}.$$

We only give a combinatorial argument for this one. We are counting the number of ways to choose an l-element subset of an $(m+n)$-element set A. Fix an m-element subset $B \subseteq A$. Any l-element subset S of A has k elements from B and $l-k$ elements from $A \setminus B$, for some $0 \le k \le l$. For a particular value of k, the number of k-element subsets of B that can be part of S is $\binom{m}{k}$ and the number of $(l-k)$-element subsets of $A \setminus B$ is $\binom{n}{l-k}$. We can now use the sum principle to sum over the possible values of k and obtain the identity. An interesting special case is

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

It follows from the Vandermonde convolution by taking $l = m = n$ and remembering that $\binom{n}{k} = \binom{n}{n-k}$.

10.2 Binomial theorem

Theorem 10.2.1. *For $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$,*

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$

Proof. By induction on n. When $n = 0$ both sides evaluate to 1. Assume the claim holds for $n = m$ and consider the case $n = m + 1$.

$$
\begin{aligned}
(x+y)^{m+1} &= (x+y)\cdot(x+y)^m && (10.1)\\
&= (x+y)\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} && (10.2)\\
&= x\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} + y\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} && (10.3)\\
&= \sum_{k=0}^{m}\binom{m}{k}x^{k+1} y^{m-k} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m+1-k} && (10.4)\\
&= \sum_{k=1}^{m+1}\binom{m}{k-1}x^{k} y^{m+1-k} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m+1-k} && (10.5)\\
&= \left(x^{m+1} + \sum_{k=1}^{m}\binom{m}{k-1}x^{k} y^{m+1-k}\right) + \left(y^{m+1} + \sum_{k=1}^{m}\binom{m}{k}x^k y^{m+1-k}\right) && (10.6)\\
&= x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\left(\binom{m}{k-1} + \binom{m}{k}\right)x^k y^{m+1-k} && (10.7)\\
&= x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}x^k y^{m+1-k} && (10.8)\\
&= \sum_{k=0}^{m+1}\binom{m+1}{k}x^k y^{m+1-k}. && (10.9)
\end{aligned}
$$

Here (5) follows from (4) by noting that

$$\sum_{k=0}^{m} f(k) = \sum_{k=1}^{m+1} f(k-1)$$

and (8) follows from (7) by Pascal's rule. The other steps are simple algebraic manipulation. This completes the proof by induction. $\square$

The binomial theorem can be used to immediately derive an identity we have seen before: By substituting $x = y = 1$ into the theorem we get

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n.$$

Here is another interesting calculation: Putting $x = -1$ and $y = 1$ yields

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

This implies

$$\sum_{k \text{ odd}} \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}.$$

This means that the number of odd-size subsets of an n-element set A is the same as the number of even-size subsets, and equals 2^{n-1}. This can be proved by a combinatorial argument as follows: Fix an element $a \in A$ and note that the number of subsets of $A \setminus \{a\}$ is 2^{n-1}. There is a bijective map between subsets of $A \setminus \{a\}$ and odd-size subsets of A, as follows: Map an odd-sized subset of $A \setminus \{a\}$ to itself, and map an even-sized subset $B \subseteq A \setminus \{a\}$ to $B \cup \{a\}$. Observe that this is a bijection and conclude that the number of odd-sized subsets of A is 2^{n-1}. Even-size subsets can be treated similarly, or by noting that their number is 2^n minus the number of odd-size ones.